The value of the test rests upon the facility with which the multiple of 21 is created, inasmuch as to multiply by 2 is a mental process much more surely within the mind of a child than dividing by 7.

2. A test for the divisibility by 13 is to be found by a similar process and upon the same principles—being based on the fact that 91 is a multiple of 7. Multiply the unit figure by 9 and find the difference between the product and the number without its unit figure. Thus, in 1183, $9\times3=27$, $118-27=91$, and in 91, $9\times1=9$, $9-9=0$. For 325, $9\times5=45$, $45-32=13$.

3. Likewise for 17; multiply by 5, since 51 is a multiple of 17. So for 595, $5\times5=25$; $59-24=34$. For 2244, $5\times4=20$, $224-20=204$; $5\times4=20$, $20-20=0$.

NOTE ON THE EVOLUTE OF AN ALGEBRAIC CURVE.

By A. H. WILSON, Instructor of Mathematics, University of Illinois.

The following method of forming the evolute of an algebraic curve may be of interest.

Let $f(x, y)=\varphi$ represent the curve, and $y-y_1=l(x-x_1)$ its normal at the point (x_1y_1) on the curve, l being a function of x_1 and y_1. The elimination of x_1 (or y_1) between $f(x_1, y_1)=0$ and $\beta-y_1=l(\alpha-x_1)$, gives an equation

$$\varphi(y_1)=0 \text{ (or } \psi(x_1)=0),$$

whose roots are the ordinates (or the abscissas) of the points on the curve the normals at which pass through the point (α, β).

The evolute may be regarded as the locus of points from which two of the normals through (α, β) to the curve are coincident; and hence the equation of the evolute is the relation between α and β obtained by setting equal to zero the discriminant of $\varphi=0$ (or $\psi=0$).

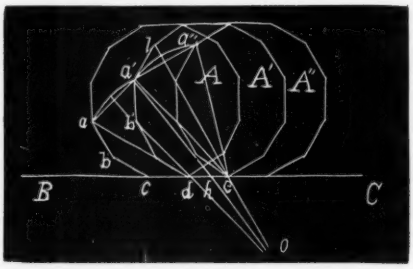
The application of the method is obviously very limited.

DETERMINATION OF THE RADIUS OF CURVATURE OF THE CYCLOID WITHOUT THE AID OF THE CALCULUS.

By FREDERIC R. HONEY, Hartford, Conn.

Let A represent any regular polygon. If we roll it along the straight line BC into the positions A', A'', ... bringing each side in succession into coinci-

dence with it, any angular point as a, will trace a series of arcs of circles aa', $a'a''$, ... whose centers will be on the line BC. The distance between two consecutive centers will be equal to the side of the polygon. The arc aa' will be described with the radius da; $a'a''$ with the radius ea', ...



The center o of an arc $aa'a''$ which will pass through the points a, a', a'', will be at the intersection of the bisector of the angle ada' and the bisector of the angle $a'ea''$. Since the polygon is inscriptible, the angle $a'el$ is equal to the alternate angle $ea'd$. Therefore, leo is parallel to $a'd$. Similarly, $b'do$ is parallel to $a'e$. Therefore, $a'eod$ is a parallelogram. Its diagonal oa', the radius with which the arc $aa'a''$ is described, is bisected at h.

The above demonstration is applicable to a regular polygon with *any* number of sides. We will now suppose that the number is increased. The length of the side diminishes, and the points d, h, and e approach each other. At the limit, when the polygon becomes a circle, they coalesce, and ha' is the normal. The broken curved line $aa'a''$..... becomes a cycloid, and the radius of curvature $oa'=2ha'$.

REMARKS ON DEFINITIONS IN TEXT-BOOKS ON GEOMETRY.

By G. W. GREENWOOD, M. A., McKendree College.

What is a circle? A common definition is that it is a plane figure—or a portion of a plane—bounded by a curved line, every point of which is equally distant from a point *within*, called the center. This gives us the impression that a circle is a disc, whereas in more advanced work it is regarded, with other conic sections, merely as a plane curve. It would be better if it were so defined in elementary texts. When we turn, however, to some texts which define it thus, we find something like this: 'A circle is a plane *closed line*, such that all straight lines joining *any* point on this line to a *certain point within* the figure, are equal.'

I shall endeavor to show that such a definition, like the more common one first given, is by no means logical. For this purpose, let us compare them with the following, which, while not altogether free from objections, is, I believe, logical: The locus of points in a plane, at congruent distances from a fixed point in the plane, is a circle. It will be noticed here that nothing is stated or implied concerning the form of the locus or any other properties save the one stated. The word *within* in the earlier definition is unnecessary unless, like a vermiform appendix, it indicates the evolution of this definition from the one first

quoted. But in addition to this we see that the definition assumes that the circle divides its plane into at least two parts, one of which is *within*. We are then supplied with other information, such as, that *since* a circle is a *closed curve*, a line from any point *within* to any point *without* the circle *intersects* the circle at least once. This assumes, among other things, that the locus is continuous; yet how all these conclusions are to be deduced from the definition, it would be difficult to state.

Most of the preliminary data in the case of circles, as given in current texts, is based entirely on our visual perception of the figure, with no attempt at deducing it from the definition; none of their off-hand statements are more evident at a glance than the fact that a straight line cannot "intersect" a circle in more than two points, yet a capricious sense of logical rigor requires that this assertion be demonstrated at length.

The premature definition of secants, tangents, tangent circles, etc., in current texts, tends to dull the sense of logical order.

We are not surprised, when the definition given of a circle is similar to one of the two quoted, to find many fallacies in their treatment of these figures. For example, consider the following: 'The line joining a point to the center of a circle is less than, or greater than, a radius, according as the point is within, or without, the circle. Let P be a point within, and Q a point without, a circle whose center is O. *Extend* OP to *meet* the circle in A, and *let* OQ *cut* the circle in C.' These statements are the very facts we wish to prove; namely, that C lies between O and Q, and that A does not lie between O and P.

Generally, with no consideration of the relative magnitudes of arcs of a circle, we are told that a chord subtends two arcs, of which the *lesser* is meant unless otherwise stated. Most of the subject as usually treated is equally vulnerable.

ON KINEMATIC GEOMETRY.—A NEW INVERSOR.

By JOHN J. QUINN, Warren, Pa.

The Theory of Inversion, an important method of investigation especially in many problems of mathematical physics, is one of the distinctive features of modern geometry. It is a method of transformation whereby one surface, or curve, is represented by another surface or curve in a manner that insures similarity in the most minute details. In general, the inverse of a sphere is a sphere; and the inverse of a circle is a circle. In order that two points be in inverse relation it is necessary that they fulfil two conditions:

1. *They must be always collinear with a fixed point.*
2. *The product of their distances from the fixed point must be constant.*

By harnessing this principle it is possible to draw a line that is mathe-

matically straight. This can be done by a linkwork so constructed that when the whole system is moved there will be three points, as O, P, and Q collinear in every position, and in the relation $OP \times OQ$=constant.

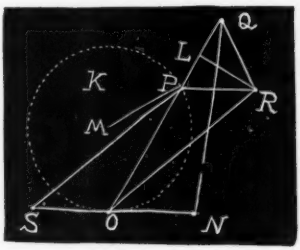
A linkwork that will describe the inverse of a given curve or surface is termed an Inversor.

The inverse points as O, P, and Q, in an inversion are called the foci.

THEOREM. *Let the side OS of the parallelogram $ORPS$ be extended to N, making ON equal to OS, and located so that NQ equals OR, and RQ equals PR. Then if this figure be embodied in a linkage:*

1. *The points O, P, and Q, are always collinear.*
2. *The points O, P, and Q, are inverse points.*
3. *If the focus O of the linkage be pivoted to a fixed base, and the focus P be constrained to move in a circle through O, the focus Q will describe a straight line.*

Given: $SP = OR = NQ$; $SO = ON = PR = RQ$.



Proof 1. $\triangle ONR \cong \triangle QNR$.

But $OP \parallel NR$.

$\therefore$ The points O, P, and Q are collinear.

Proof 2. Draw RL perpendicular to OPQ. Then $OP = OL - LP = \sqrt{(OR^2 - RL^2)} - \sqrt{(PR^2 - RL^2)}$; $OQ = OL + LP = \sqrt{(OR^2 - RL^2)} + \sqrt{(PR^2 - RL^2)}$.

$\therefore OP.OQ = OR^2 - PR^2$ = a constant.

$\therefore$ The points O, P, and Q are inverse points.

Proof 3. $\because$ O, P, and Q are inverse points, and since a straight line is the inverse of a circle through the pole of inversion,

$\therefore$ The line QV is a straight line, the inverse of the circle K. Q. E. D.

SIX PROPOSITIONS ON PRIME NUMBERS.

By R. D. CARMICHAEL, Hartselle, Ala.

PROPOSITION A. *If p is an odd prime,* $(1.2.3.....\frac{p-1}{2})^2 + (-1)^{\frac{1}{2}(p-1)} \equiv 0 \pmod{p}$.

By Wilson's theorem, if p is prime

$$1.2.3.....(p-1) + 1 \equiv 0 \pmod{p} \qquad (1).$$

This may be written

$$(1.2.3.....\frac{p-3}{2}.\frac{p-1}{2})(p-\frac{p-1}{2})(p-\frac{p-3}{2}).....(p-2)(p-1)+1 \equiv 0 \pmod{p} \qquad (2).$$

Expanding and casting out all terms containing p as a factor (p being an odd prime),

$$(1.2.3.....\frac{p-1}{2})^2+(-1)^{\frac{1}{2}(p-1)}\equiv 0 \pmod{p} \qquad (3).$$

PROPOSITION B. *If* $(1.2.3.....\frac{p-1}{2})^2+(-1)^{\frac{1}{2}(p-1)}\equiv 0$ *(mod* p*),* p *is prime.*

It is evident that we may revert from (3) to (2) by re-inserting the cast out terms containing p as a factor. (2) is but another form of (1). Hence, we have only to verify the well-known theorem that p is a prime, if

$$1.2.3.....(p-1)+1\equiv 0 \pmod{p} \qquad (4)$$

For any value of p *not* prime, except $p=4$, it is easily shown that

$$1.2.3.....(p-1)\equiv 0 \pmod{p} \qquad (5).$$

Hence, when (4) is true, p is prime.

PROPOSITION C. *If* $4n+1$ *is prime, it may be expressed as the sum of two parts* r *and* s *such that* $r^2+1\equiv 0$, $s^2+1\equiv 0$, *and* $rs-1\equiv 0$ *(mod* p*).*

For p a prime of the form $4n+1$, (3) may be written

$$(1.2.3.....2n)^2+1\equiv 0 \pmod{4n+1} \qquad (6).$$

This may be written in either of the forms,

$$[l(4n+1)+r]^2+1\equiv 0 \pmod{4n+1} \qquad (7),$$
$$[(l+1)(4n+1)-s]^2+1\equiv 0 \pmod{4n+1} \qquad (8),$$

where l is an integer and $r+s=4n+1$. Hence, expanding and casting out the terms containing $4n+1$ as a factor, we get

$$r^2+1\equiv 0,\ s^2+1\equiv 0 \pmod{4n+1} \qquad (9),$$

from which easily follsws $rs-1\equiv 0$.

PROPOSITION D. *If* $a+1$ *and* $2a+1$ *are both primes,* $(1.2.3.....a)^4-1\equiv 0$ *[mod* $(a+1)(2a+1)$*].*

By Wilson's theorem,

$$1.2.3.....a+1\equiv 0 \pmod{a+1} \qquad (10).$$
$$\therefore (1.2.3.....a)^4-1\equiv 0 \pmod{a+1} \qquad (11).$$

By Proposition A above,

$$(1.2.3.....a)^2+(-1)^a\equiv 0 \pmod{2a+1} \qquad (12).$$

$$\therefore (1.2.3.....a)^4-1\equiv 0 \pmod{2a+1} \qquad (13).$$

Since $a+1$ and $2a+1$ are prime to each other, (11) and (13) lead to the theorem stated.

PROPOSITION E. *If* $(1.2.3.....a)^4\equiv 0$ [*mod* $(a+1)(2a+1)$], $a+1$ and $2a+1$ *are both prime.*

We have to show that $a+1$ and $2a+1$ are prime when (11) and (13) hold.

If $a+1=4$, (11) does not hold. For all other values of $a+1$ *not* prime, $1.2.3.....a\equiv 0 \pmod{a+1}$. Hence, when $a+1$ is not prime, (11) does not hold. Therefore, $a+1$ is prime under the given condition.

If $2a+1$ is *not* prime, $1.2.3.....2a\equiv 0 \pmod{2a+1}$.

$$\therefore (1.2.3.....(a-1)a.[(2a+1)-a][(2a+1)-(a+1)].....$$
$$.....[(2a+1)-2][(2a+1)-1]\equiv 0 \pmod{2a+1}.$$

Expanding and casting out terms containing $2a+1$ as a factor, we get

$$(1.2.3.....a)^2\equiv 0 \pmod{2a+1} \qquad (14),$$

if $2a+1$ is *not* prime. Hence, when (13) holds, as (14) does not then hold, $2a+1$ is odd.

PROPOSITION F. *If* $a+1$ *and* $2a+1$ *are both prime,* $(a+1)(2a+1)$ *may be expressed as the sum of* r *and* s *such that* $r^4-1\equiv 0$, $s^4-1\equiv 0$, *and* $r^2s^2-1\equiv 0$ [*mod* $(a+1)(2a+1)$].

The demonstration depends upon Proposition D above, and is similar in method to that of Proposition C. It may easily be supplied by the reader.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

No. 223 was also solved by L. E. Newcomb.

Mr. L. S. Shively calls attention to the fact that a solution of No. 225 is given in C. Smith's "A Treatise on Algebra," page 183, Ex. 4.

226. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Find the real roots of the system

$$x^2+w^2+v^2=a^2, \quad vw+u(y+z)=bc,$$
$$w^2+y^2+u^2=b^2, \quad wu+v(z+x)=ca,$$
$$v^2+u^2+z^2=c^2, \quad uv+w(x+y)=ab.$$

Solution by A. H. HOLMES, Brunswick, Me.

$x^2+w^2+v^2=a^2$ .....(1), $vw+u(y+z)=bc$.....(4),
$w^2+y^2+u^2=b^2$ .....(2), $wu+v(z+x)=ca$.....(5),
$v^2+u^2+z^2=a^2$ .....(3), $uv+w(x+y)=ab$.....(6).

From (2), (3), and (4),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[vw+u(y+z)]^2.$$
$$\therefore (yv-uw)^2+(uv-zw)^2+(u^2-yz)^2=0.$$
$$\therefore yv=uw \text{ .....(7)}; \; uv=zw \text{.....(8)}; \; u^2=yz \text{ .....(9)}.$$

From (1), (3), and (5),

$$(x^2+w^2+v^2)(v^2+u^2+z^2)=[wu+v(z+x)]^2.$$
$$\therefore v^2=xz \text{.....(10)}; \; vw=xu \text{.....(11)}.$$

From (2), (3), and (6),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[uv+w(x+y)]^2.$$
$$\therefore w^2=xy \text{.....(12)}.$$

Substituting xy and xz for w^2 and v^2 in (1), xy and yz for w^2 and u^2 in (2), and xz and yz for v^2 and u^2 in (3), adding the three equations and extracting square root, we obtain $x+y+z=\sqrt{(a^2+b^2+c^2)}$.

Substituting xu for vw in (4), $u=\dfrac{bc}{\sqrt{(a^2+b^2+c^2)}}$.

Similarly, $v=\dfrac{ac}{\sqrt{(a^2+b^2+c^2)}}$, $w=\dfrac{ab}{\sqrt{(a^2+b^2+c^2)}}$,

$$x=\frac{a^2}{\sqrt{(a^2+b^2+c^2)}}, \quad y=\frac{b^2}{\sqrt{(a^2+b^2+c^2)}}, \quad z=\frac{c^2}{\sqrt{(a^2+b^2+c^2)}}.$$

227. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Solve $x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3=11$; $x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^2y^3+2x^2y^4+4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4+x^2y^5+x^3y^5=30$.

Solution by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Put $X=(x+y+xy+x^2y+xy^2)$, and $Y=(x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3)$; then the given equations become, respectively,

$$X+Y=11 \text{.....}(\alpha), \text{ and}$$
$$XY=30 \text{.....}(\beta).$$

$\therefore X=6$, or 5; and $Y=5$, or 6.

By putting $X_1=(x+y+xy)$, and $Y_1=(x^2y+xy^2)$, the expressions represented by X and Y give, respectively,

$$X_1+Y_1=6, \text{ or } 5 \ldots\ldots (\alpha_1), \text{ and}$$
$$X_1 Y_1=5, \text{ or } 6 \ldots\ldots (\beta_1).$$

$\therefore X_1=5$, 1, 3, or 2; and $Y_1=1$, 5, 2, or 3.

By putting $X_2=(x+y)$ and $Y_2=(xy)$, the expressions represented by X_1 and Y_1 give, respectively,

$$X_2+Y_2=5, 1, 3, \text{ or } 2 \ldots\ldots (\alpha_2), \text{ and}$$
$$X_2 Y_2=1, 5, 2, \text{ or } 3 \ldots\ldots (\beta_2).$$

$\therefore x+y=\frac{1}{2}(5\pm\sqrt{21})$, $\frac{1}{2}(1\pm\sqrt{-19})$, 2 or 1, or $1\pm\sqrt{-2}$; and
$xy=\frac{1}{2}(5\mp\sqrt{21})$, $\frac{1}{2}(1\mp\sqrt{-19})$, 1 or 2, or $1\mp\sqrt{-2}$.

Solving these *eight* simultaneous equations, we have the *sixteen* values of x and y.

Also solved by J. Scheffer, Henry Heaton, G. B. M. Zerr, and A. H. Holmes.

GEOMETRY.

254. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the cartesian equation to a line that is both tangent and normal to the cardioid.

[No solution of this problem has been received.]

255. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the envelope of the straight line that connects the extremities of two conjugate diameters of an ellipse.

I. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

If we project the ellipse into a circle, the conjugate diameters are projected into perpendicular diameters of the circle, whose chord envelopes a concentric circle. Hence in the original figure the chord envelopes a similar, and similarly situated concentric ellipse.

II. Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

The equation to the line is, with usual notation,

$$\frac{x}{a}\cos(\varphi+\tfrac{1}{4}\pi)+\frac{y}{b}\sin(\varphi+\tfrac{1}{4}\pi)=\cos\tfrac{1}{4}\pi, \text{ or } \left(\frac{x}{a}+\frac{y}{b}\right)\cos\varphi-\left(\frac{x}{a}-\frac{y}{b}\right)\sin\varphi=1,$$

$$i.e., \left(\frac{x}{a}+\frac{y}{b}-1\right)-2\left(\frac{x}{a}-\frac{y}{b}\right)\tan\tfrac{1}{2}\varphi-\left(\frac{x}{a}+\frac{y}{b}+1\right)\tan^2\tfrac{1}{2}\varphi=0.$$

$\therefore$ The equation to the envelope is

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2+\left(\frac{x}{a}+\frac{y}{b}+1\right)\left(\frac{x}{a}+\frac{y}{b}-1\right)=0, \text{ i. e., } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}.$$

Solved similarly by J. Scheffer.

256. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

The bisectors of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.; FREDERIC R. HONEY, Ph. D., Trinity College, Hartford, Conn.; R. D. CARMICHAEL, Hartselle, Ala., and A. H. HOLMES, Brunswick, Me.

Denote the quadrilateral by $ABCD$, and the bisectors of the angles A, B, C, D by a, b, c, d, respectively. Suppose AB, DC intersect at O; a, d at X; and b, c at Y. We can easily show that, if B lies between A and O,

$$\angle AXD=\tfrac{1}{2}\pi+\tfrac{1}{2}\angle AOD;$$
$$\angle BYC=\tfrac{1}{2}\pi-\tfrac{1}{2}\angle AOD.$$

Hence, a pair of opposite angles of the quadrilateral whose sides are, consecutively, a, b, c, d, are supplementary, and it is therefore cyclic.

*** The problem admits of the following interesting extension: If a', b', c', d' are the bisectors, respectively, of the exterior angles A, B, C, D, the six quadrilaterals whose consecutive sides are, respectively, $(abcd)$, $(abc'd')$, $(bcd'a')$, $(cda'b')$, $(dab'c')$, $(a'b'c'd')$, are cyclic. It is necessary that no two sides of the given figure be parallel. GREENWOOD.

Also solved by F. P. Matz, J. Scheffer, G. B. M. Zerr, and W. J. Greenstreet. A. H. Holmes also solved No. 252.

CALCULUS.

194. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Show that the volume of the solid generated by the revolution of a segment of a circle, less than a semi-circle, about the diameter parallel to its chord, is equal to that of a sphere having a diameter equal to the chord; and hence that the volume is independent of the magnitude of the original circle, the length of the chord being known.

Solution by W. L. TRYON, Ithaca, N. Y.; R. D. CARMICHAEL, Hartselle, Ala.; and W. J. GREENSTREET, Editor of The Mathematical Gazette, Stroud, England.

Let the chord be of length $2b$ in circle of radius a; and take as axes the parallel diameter and the perpendicular bisector of the chord. The distance of the chord from the center is $\sqrt{(a^2-b^2)}$. The required volume is

$$V=2\int_0^b(a^2-x^2)dx-2\pi b(a^2-b^2)=2\pi b(a^2-\tfrac{1}{3}b^2)-2\pi b(a^2-b^2)=\tfrac{4}{3}\pi b^3.$$

This is independent of a, and is equal to the volume of a sphere of diameter $2b$.

Also solved by C. Hornung, J. Scheffer, A. H. Holmes, and the Proposer.

MECHANICS.

173. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A squirrel is in a cylindrical cage and oscillating with it about its axis which is horizontal. At the instant when he is at the highest point of the oscillation, he leaps to the opposite extremity of the diameter and arrives there at the same instant as the point at which he left. Determine his leap completely.

[No satisfactory solution has been received.]

174. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

By what per cent. is the striking force of a hailstone increased in falling 1000 feet through a stratum of atmosphere moving uniformly eastward at the rate of 60 miles an hour?

Solution by the PROPOSER.

Let S_1 = the *striking-force* of the hail-stone falling 1000 feet, and $S_2 = mS_1$ = that caused by the horizontal motion of the atmosphere during the time required to fall 1000 feet; then the resultant striking-force is $\sqrt{(1+m^2)}S_1$. Assuming a mass of ice suitable for a hail-stone, we can easily calculate m.

The required percentage of increase in striking-force becomes

$$I=\left(\frac{\sqrt{(1+m^2)}-1}{1}\right) \text{ of } 100\%.$$

175. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Oklahoma.

A cylinder descends down a plane, the inclination of which to the horizon is α, unwrapping a fine string fixed at the highest point of the plane. Find the angle through which the plane must be depressed in order that a sphere, descending under like circumstances, may experience the same acceleration.

Solution by G. W. GREENWOOD, M. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let ω be the angular velocity, and v the velocity parallel to the plane, after rolling a distance l from the highest point. Equating the *vis viva* of the body to twice the work done by gravity, we get, in the case of the cylinder,

$$\tfrac{1}{2}ma^2\omega^2+mv^2=2mgl\sin\alpha,$$

where a is the radius of the cylinder.

$\therefore 3v^2=4gl\sin\alpha$, since $a\omega=v$.

In the case of the sphere we get

$$\tfrac{2}{5}m'a'\omega^2+m'v^2=2mgl\sin(\alpha-\theta), \textit{ i. e., } 7v^2=10gl\sin(\alpha-\theta),$$

where θ is the amount of depression of the plane.

$\therefore 14\sin\alpha=15\sin(\alpha-\theta)$, or $\theta=\alpha-\sin^{-1}(\tfrac{14}{15}\sin\alpha)$.

DIOPHANTINE ANALYSIS.

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find (1) three square numbers whose sum is a cube; (2) three cube numbers whose sum is a square.

I. Remark by J. SCHEFFER, Hagerstown, Md.

(1) The sum of the three squares 9, 16, 100 gives the cube number 125, hence $9n^6+16n^6+100n^6=125n^6=(5n^2)^3$, where n=any integer. There are doubtless other sets of numbers satisfying this condition.

(2) Since the sum of the cubes of three Pythagorean numbers is a square, one set of numbers satisfying this condition is: $27\times6^{6n-3}$, $64\times6^{6n-3}$, $125\times6^{6n-3}$, whose sum$=6^{6n}$.

II. Remark by the PROPOSER.

(1) Since $5^2+6^2+8^2=5^3$, the numbers $5n$, $6n$, $8n$, n=integer, satisfy the required condition.

(2) Since $1^3+2^3+3^3=6^2$ the numbers n^3, $8n^3$, $27n^3$, n=integer, satisfy the required condition.

Also solved by G. B. M. Zerr, and A. H. Holmes.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

What *values* of x will make $E=\dfrac{(x+7)(x+5)}{(x-7)(x-5)}$ represent square numbers?

Solution by G. B. M. ZERR.

$$E=\frac{(x+7)(x+5)}{(x-7)(x-5)}=\frac{(1+7/x)(1+5/x)}{(1-7/x)(1-5/x)}.$$

Hence, two values of x which satisfy the required condition are 0, ∞.

Also solved by J. Scheffer, and the Proposer.

AVERAGE AND PROBABILITY.

158. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

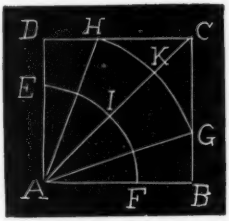
In a given square an arc is described at random the center being one of the vertices of the square. What is the probability that this arc is longer than a side of the square?

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the square; A the vertex from which the random arc is drawn; and EIF and HKG arcs whose lengths are equal to a, the length of a side of the square. Then if the random arc falls between the arcs EIF and HKG its length will exceed the length of a side of the square.

Since no law of distribution of the events is given, we will assume that the intersections of the arcs with the line AC are uniformly distributed on AC.

Hence, if the random arc intersects AC between I and K its length will exceed a, the length of a side of the square.



Hence, the required probability is

$$p=\frac{IK}{AC}=\frac{AK-AI}{AC}.$$

But, since $EIF=\frac{1}{2}\pi.AI=a$, $AI=2a/\pi$, and likewise, $AK=a/\theta$, where $\theta=\angle HAG$.

$\therefore p=\dfrac{(a/\theta-2a/\pi)}{a\sqrt{2}}=\dfrac{1}{\theta\sqrt{2}}-\dfrac{\sqrt{2}}{\pi}$, where θ may be found by the method of Double Position from the equation, $\cos(\frac{1}{4}\pi-\frac{1}{2}\theta)=\theta$.

For $\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=\dfrac{AB}{AG}=\dfrac{a}{AG}$, and $\theta.AG=HKG=a$.

Hence, $\theta=\dfrac{a}{AG}$, and therefore $\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)=\theta$.

Solving this equation we find $\theta=.95266$ radians.

$$\therefore p=\frac{1}{.95266\sqrt{2}}-\frac{\sqrt{2}}{\pi}=.2920+.$$

Also solved by F. P. Matz who distributes the events in proportion to the area between the limiting arcs and the area of the square; G. B. M. Zerr, who gets as a result by using the calculus, .0734; and Henry Heaton, who finds the mean lengths of the arcs.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

A box contains n tickets numbered from 1 to n. How many draws, on the average, will it take to draw all the numbers, each ticket being replaced before drawing again? What is the numerical result for $n=2$ and $n=6$?

Solution by W. W. LANDIS, A. M., Professor of Mathematics, Dickinson College, Carlisle, Pa.

The chance of drawing any particular number at least once in p drawings is

$$1-\left(1-\frac{1}{n}\right)^p.$$

The chance that all will be drawn in p drawings (p being, of course$>n$) is

$$\left[1-\left(1-\frac{1}{n}\right)^p\right]^n,$$

which by the conditions of the problem must equal $\frac{1}{2}$. Solving this equation for p,

$$p=\frac{\frac{1}{n}\log 2-\log(2^{1/n}-1)}{\log n-\log(n-1)}.$$

For $n=2$, $p=1.75+$, hence two drawings must be made; for $n=6$, $p=12.15+$; hence, thirteen drawings must be made.

Also solved with the same result by F. O. Whitlock, and by a different method, which seems to me to be incorrect, by S. A. Corey, Hiteman, Iowa. The problem is equivalent to that of Problem IX, page 52, of Meyer's *Wahrscheinlichkeitsrechnung*. F.

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

The year 1905 *began*, and will *end*, on a Sunday. Prove that this can not occur again until the year 2015.

Solution by WILLIAM HOOVER, Ph. D., Athens, Ohio.

The Dominical Letter for Sunday when on January 1 is A, and also when on December 31, the year being common. Those common years in the present century fulfilling the required conditions must have A for their Dominical Letter; such years are 1905, 1911, 1922, 1933, 1939, 1950, 1961, 1967, 1978, 1989, 1995, sufficient to show that the statement in the problem is not true.

Remark by Proposer. The year 2015 will begin on a Thursday.
Also solved by A. H. Holmes, Henry Heaton, G. B. M. Zerr, and G. W. Greenwood.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If an *unknown* curve be described under a constant acceleration not tending to the center and the hodograph is a cardioid, what is the unknown curve?

I. Solution by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Let r and p be the radius vector and perpendicular upon the tangent to the curve at the outer extremity of r and r'; p' the analogous lines in the hodograph, and h the double area generated by r in a unit of time.

Then by the theory of the hodograph,

$$r'=\frac{h}{p}\ldots\ldots(1),\quad p'=\frac{h}{r}\ldots\ldots(2).$$

Also, from the theory of central forces,

$$k=\frac{h^2}{p^3}\frac{dp}{dr}\ldots\ldots(3),$$

and for the cardioid,

$$p'^2=\frac{r'^3}{2a}\ldots\ldots(4).$$

Substituting r' and p' from (1) and (2) in (4),

$$h=\frac{2ap^3}{r^2}\text{.....(5).}$$

Then (5) in (3), gives

$$kr^4dr=4a^2p^2dp\text{.....(6),}$$

the differential equation to the required orbit.

Integrating (6), and supposing r and p to vanish together,

$$\frac{k}{5}r^5=a^2p^4\text{.....(7),}$$

the required orbit.

II. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

Let the equation to the hodograph be $r=2a\cos^2\frac{1}{2}\theta$. Since the acceleration in the original curve is constant the velocity of the point in the hodograph is constant, and $s=4akt$, where k is some constant; that is $kt=\sin\frac{1}{2}\theta$.

From the equation to the hodograph we have $v=2a\cos^2\frac{1}{2}\psi$, where v is the velocity in the orbit and ψ is the inclination of the tangent at that point to the initial line.

$$\therefore \frac{ds}{dt}=2a\cos^2\tfrac{1}{2}\psi=2a(1-k^2t^2).$$

$$\therefore s=2a(t-\tfrac{1}{3}k^2t^3);\ i.\ e.,\ s=\frac{2a}{k}(\sin\tfrac{1}{2}\psi-\tfrac{1}{3}\sin^3\tfrac{1}{2}\psi),$$

which is the intrinsic equation to the orbit.

Also solved by G. B. M. Zerr, S. A. Corey, and the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

234. Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Prove that $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\text{.....}=n!$

235. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

Easter Sunday, 1905, was on April 23. How often in the last one hundred years has this occurred, and when?

236. Proposed by L. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

Sum to infinity the series $\frac{n^2}{(n+1)!}$ beginning with $n=1$.

GEOMETRY.

257. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Construire un triangle équilatéral sachant qu'el doit s'appuyer par ses trois sommets sur trois circonférences concentriques données. *Rouché et Comberousse.*

258. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Mo.

Prove that the tangents to an ellipse from any external point subtend equal angles at the focus, by means of the formula $\tan \phi = (m_1 - m_2) / (1+m_1 m_2)$, where ϕ is the angle between the focal radius of either of the points of tangency and the line joining the focus and the external point, and m_1 and m_2 are the slopes of these two lines.

259. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Given three non-intersecting circles; to draw eight tangent circles, each tangent to all three of the given circles.

CALCULUS.

196. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

The shortest tangent intercepted by the axes of the ellipse to which the tangent is drawn, equals the sum of the semi-axes of the ellipse.

197. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

$$\int_0^\infty \frac{\sin mx \cos nx}{x} dx.$$

189. Proposed by M. E. GRABER, A. M., Heidelburg University, Tiffin, O.

Show that $e\int_1^\infty$, $\varepsilon^2\int_2^\infty$, ... $\varepsilon^n\int_n^\infty$, ... are integers divisible by $(p+1)!$, when the expression under the integral is $Z^p\Big[(z-1)\ldots\ldots\ldots\ldots(z-n)\Big]^{p+1} e^{-z}\,dZ$.

MECHANICS.

178. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

A weight W is drawn up a rough conical hill of height h and slope α. and

the path cuts all the lines of greatest slope at the constant angle β. Find the work done in attaining the summit.

[Problem 11, page 226, *Johnson's Theoritical Mechanies.*]

179. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the *velocity* of a body moving under an acceleration tending to the center *varies* as the radius of curvature, the body will describe a cycloid.

MISCELLANEOUS.

148. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Given $\sin 3\phi + \cos 3\phi = m$.....(1), and $\cos\phi - \sin\phi = x$.....(2), to find x extremes of m.

AVERAGE AND PROBABILITY.

163. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle.

164. Proposed by J. O. Mahoney, B. E., M. Sc., Central High School, Dallas, Texas.

If m is prime, and the numbers 0, 1, 2, 3,....., m^2-1 are placed at random in the form of a square, the probability that the square is *hyper-magic* is

$$\frac{(m-1)\,m}{(m^2-2)!}$$

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

The Chicago Section of the American Mathematical Society met in Chicago on April 29. E.

Mr. Newton Ensign, of McKendree College, a student of our well known contributor Prof. G. W. Greenwood, was awarded the Rhodes Scholarship for Illinois. He will pursue the honor mathematical course at Oxford University. E.

The Open Court Publishing Co., of Chicago, has just issued a portfolio of twelve portraits of eminent mathematicians, edited by Professor David Eugene Smith. It includes the portraits of DeCartes, Pythagoras, Archimedes, Fermat,

Leonardo of Pisa, Euclid, Leibnitz, Napier, Vieta, Newton, Thales. The portraits are printed by a photographic process, and are issued in two forms, the first on Japanese paper, the price of which is $5, and the second on plate paper, at $3. The originals are from Professor Smith's large collection. With each portrait is a biographical note introducing a brief biography. E.

Editor Finkel has been appointed to a Harrison Fellowship in the University of Pennsylvania for the year 1905-1906. The Board of Trustees of Drury College has granted him a year's leave of absence and he will spend the coming year in study in Philadelphia. Mr. O. E. Glenn, Fellow in Mathematics in the University of Pennsylvania, has been appointed Acting Professor of Mathematics and Physics during his absence.

Dr. Edward Kasner has been promoted to an instructorship in Mathematics in Barnard College, Columbia University.

Professor E. B. VanVleck, of Wesleyan University has received leave of absence and will spend the year abroad.

Professor L. W. Dowling, of the University of Wisconsin, has been granted leave of absence and will spend the coming year mainly in Italy.

Mr. W. H. Roever has been appointed Instructor in Mathematics at the Massachusetts Institute of Technology.

BOOKS AND PERIODICALS.

The Secret of the Circle and the Square. By J. C. Willmon. 12 mo. Cloth, 30 pages. Los Angeles, California: Published by the Author.

This little volume should not be consigned to the junk-pile of literature treating on successful (?) circle-squaring and angle trisecting, as one might infer from its title. The author states in his preface that it is his intention "to demonstrate the possibility of constructing a straight line equal to any given arc of a circle, and through this problem to construct a square equal in area to any circle and a circle equal in area to any square, with solutions of kindred geometrical problems." The constructions are simple, though only approximate, but may, by the method given, be carried out to any desired degree of accuraey. B. F. F.

Plane and Spherical Trigonometry. By P. A. Lambert, Assistant Professor of Mathematics, Lehigh University, and H. A. Foering, Head Master of Bethlehem Preparatory School. 8vo. Cloth, 104 pages. Price, 60 cents. New York: The Macmillan Co.

This text-book gives about the amount of material required of the average college student. The definitions of the functions are given by means of projection, a feature that

is rapidly gaining favor among teachers. The book contains neither tables nor answers to the problems. B. F. F.

Euclid's Parallel Postulate, Its Nature, Validity, and Place in Geometrical Systems. By John William Withers, Ph. D., Principal of the Yeatman High School, St. Louis, Mo. 8vo. Cloth, vii + 192 pages. Chicago: The Open Court Publishing Co.

This is the author's thesis presented to the Philosophical Faculty of Yale University for the Degree of Doctor of Philosophy. The author discusses the subject quite fully in seven chapters. In the first chapter he traces the pre-Lobatchevskian struggle with the parallel postulate; in the second, he considers the discovery and development of non-Euclidean systems; in the third, he notes the general Orientation of the problem; in the fourth, the psychology of the parallel postulate and its kindred conceptions are treated; in the fifth, the nature and validity of the parallel postulate is set forth; and in the sixth, the resulting implications as to the nature of space are noted. The seventh chapter is given a fairly complete Bibliography of non-Euclidean geometry. B. F. F.

The New Knowledge. A popular account of the New Physics and New Chemistry in relation to the New Theory of Matter. By Robert Kennedy Duncan, Professor of Chemistry in Washington and Jefferson College. 8 vo. Cloth, xviii+263 pages. Price, $2.00. New York: A. S. Barnes & Co.

The author of this book has given a very remarkable presentation of the whole field of recent discoveries, ranging from the atoms of the elements, corpuscles, the various rays, radioactivity, and inter-atomic energy, to inorganic evolution and cosmical problems. It is written in a simple though clear style, and, while dealing with the profoundest subjects that ever engaged the attention of man, requires no more of its readers than a love of contemporary natural science and a high school education. The work is one that every person interested in the progress and achievements of the human race, the marvelous inventions and discoveries, will want to read and reread. B. F. F.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. James, Ph. D., Instructor in Mathematics and Astronomy, Washington University, St. Louis. Small 8vo. Cloth, xii+176 pages, 39 figures. Price, $2.00. New York: John Wiley & Sons.

This book is intended for those students who wish to continue the study of Mechanics beyond an elementary course, and is meant to serve as an introduction to advanced treatises. For this reason applications have been almost entirely omitted, to the detriment, it seems to me, of the usefulness of the book, since nothing so firmly clinches a principle as a well selected application of that principle. Problems requiring a knowledge of the calculus and elementary differential equations have been either entirely omitted or approximate solutions only have been given. The book closes with an elementary and concise treatment of Constrained Motions on the Earth's Surface. B. F. F.

Graphic Algebra for Secondary Schools. By H. B. Newson, Ph. D., Associate Professor of Mathematics in the University of Kansas. Pamphlet, 19 pages. Boston and Chicago. Ginn & Co.

This pamphlet is the author's interpretation of the recent Committee of the American Mathematical Association's recommendation both as to kind and amount of graphical work to be used in illustrations and in connection with the solutions of equations. F.

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THE BALLISTIC PROBLEM.

By S. A. COREY.

The problem of the motion of a projectile through a resisting medium, such as air, is discussed at some length by Gilman in the April number of the *Annals of Mathematics*. Of the three methods there given the most accurate is exceedingly laborious. The following method, based on Stirling's formula, is very accurate, simple, and involves but little labor.

It is known that Stirling's formula may be used to develop the following formula:*

$$f(a+x)=f(a)+\frac{x}{m}\left[\frac{f'(a+x)+f'(a)}{2}+f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)\right.$$

$$\left.+f'\left(a+\frac{3x}{m}\right)+\ldots\ldots+f'\left(a+\frac{m-1}{m}x\right)\right]-\frac{B_1 x^2}{m^2.2!}\left[f'(a+x)-f'(a)\right]$$

$$+\frac{B_2 x^4}{m^4.4!}\left[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\right]-\ldots\ldots$$

$$+(-1)^n\frac{B_n x^{2n}}{m^m.(2n)!}\left[f^{(2n)}(a+x)-f^{(2n)}(a)\right]+\ldots\ldots(A),$$

B_1, B_2, B_3, etc., being Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc.

Using the first two terms only and taking $m=3$, (A) becomes

*See *Annals of Mathematics*, Vol. 5, No. 4, July, 1904.